

A
Generalization
of Mordell to
Ternary
Quadratic
Forms

Sarah
Blackwell,
Gabe Durham,
Tiffany Treece

Introduction
and History

Outline of
Mordell's
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A Generalization of Mordell to Ternary Quadratic Forms

Sarah Blackwell, Gabe Durham, Tiffany Treece
joint with Katherine Thompson

The Wake Forest/Davidson Experience in Number Theory Research

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Acknowledgements

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History of the Sum of Three Squares

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- Diophantus asked for a given $m \in \mathbb{Z}$, when do there exist $a, x, y, z \in \mathbb{Z}$ such that $m = x^2 + y^2 + z^2$ and $x + a, y + a$ and $z + a$ are all perfect squares.
- The case where $a = 0$ eventually became of special interest to number theorists; i.e., when is a number the sum of three squares?

History of the Sum of Three Squares

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- In 1636 Fermat stated that no integer of the form $8k + 7$ is a sum of three squares.
- In 1798 Legendre announced a proof of the theorem.
- As a result the statement is known as: “Legendre’s Three Squares Theorem.”
- The theorem states that if $m \in \mathbb{Z}$ and m is not of the form $4^k(8\ell + 7)$ with $k, \ell \in \mathbb{Z}$, then there exists $x, y, z \in \mathbb{Z}$ such that $m = x^2 + y^2 + z^2$.

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- In 1801 Gauss provided a proof of the theorem in his famous work *Disquisitiones Arithmeticae*.
- He did so by giving a formula for the number of representations of an integer as a sum of three squares.

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Other proofs given for the theorem:

- 1850 - Dirichlet - Uses his theorem of primes in an arithmetic progression.
- 1957 - Ankeny - Uses Geometry of Numbers.
- 1960 - Mordell - Refines Ankeny's techniques.

Mordell's Proof - Sum of Three Squares

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- Mordell's proof of the three squares theorem begins by introducing a polynomial $f(x, y, z)$ with integer coefficients given by:

$$mf(x, y, z) = (Ax + By + mz)^2 + (ax^2 + 2hxy + by^2).$$

- The proof then uses a GoN result of Gauss to show that there are integers x, y, z such that $f(x, y, z) = 1$.
- Because of this result and because all the cross terms of the polynomial are even, he is able to complete the square and use a change of variables to conclude that $f(x, y, z)$ equals the sum of three squares.
- By construction, there exist x, y, z such that $f(x, y, z) = m$.

Mordell's Proof - Sum of Three Squares

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- We can show what integers m can be written as the sum of three squares as long as all of $A, B, a, b,$ and h exist.
- The selection must satisfy the conditions:
 $A^2 + a \equiv 0, B^2 + b \equiv 0, 2AB + 2h \equiv 0 \pmod{m}$ set by the construction of the polynomial.
- Additionally, we must satisfy the condition $ab - h^2 = m$.
- The proof uses quadratic reciprocity and Dirichlet's theorem of primes in an arithmetic progression to achieve this.

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The question regarding the sum of three squares is a special case of a much broader class of problems.

Definitions

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- In our research we only considered integral positive definite ternary quadratic forms.
- An integral ternary quadratic form Q is a homogeneous polynomial of degree 2, where:

$$Q : \mathbb{Z}^3 \rightarrow \mathbb{Z} \\ (x_1, x_2, x_3) \mapsto \sum_{1 \leq i < j \leq 3} a_{ij} x_i x_j$$

where $a_{ij} \in \mathbb{Z} \forall i, j$.

- If we represent the quadratic form Q by a symmetric matrix M , then the determinant of the form Q is the determinant of the matrix M .

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- A quadratic form Q is positive definite if:
 - (i) $Q(\vec{x}) = 0 \Leftrightarrow \vec{x} = \vec{0}$
 - (ii) $\forall \vec{x} \neq \vec{0}, Q(\vec{x}) > 0$.
- We say that an integer m is represented by a ternary quadratic form, Q , if $\exists \vec{x} \in \mathbb{Z}^3$ such that $Q(\vec{x}) = m$.
- Given two forms Q and P with their associated matrices M_Q and M_P , we say $Q \sim P$ if $\exists A \in GL_n(\mathbb{Z})$ with $A^t M_Q A = M_P$.

Ramanujan-Dickson Forms

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- We wanted to see if Mordell's argument could generalize to quadratic forms other than just the sum of three squares. We first chose to study the Ramanujan-Dickson forms.
- In the early 20th century Ramanujan wanted to classify the universal diagonal quaternary quadratic forms, using the technique of escalation.
- Dickson proved exactly which numbers were represented by these escalators in 1927 by using reduction theory.

Our Results

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Theorem

- (a) A positive integer m is represented by $x^2 + y^2 + 2z^2$ if and only if $m \neq 4^k(16\ell + 14)$.
- (b) A positive integer m is represented by $x^2 + y^2 + 3z^2$ if and only if $m \neq 9^k(9\ell + 6)$.
- (c) A positive integer m is represented by $x^2 + 2y^2 + 2z^2$ if and only if $m \neq 4^k(8\ell + 7)$.
- (d) A positive integer m is represented by $x^2 + 2y^2 + 3z^2$ if and only if $m \neq 4^k(16\ell + 10)$.
- (e) A positive integer m is represented by $x^2 + 2y^2 + 4z^2$ if and only if $m \neq 4^k(16\ell + 14)$.
- (f) A positive integer m is represented by $x^2 + 2y^2 + 5z^2$ if and only if $m \neq 25^k(25\ell \pm 10)$.

Our Results

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Additionally we used our method on forms of determinants 5 and 6:

Theorem

- (a) *A positive integer m is represented by $x^2 + y^2 + 5z^2$ if and only if $m \neq 4^k(8\ell + 3)$.*
- (b) *A positive integer m is represented by $x^2 + 2y^2 + 2yz + 3z^2$ if and only if $m \neq 25^k(25\ell \pm 5)$.*

Theorem

A positive integer m is represented by $x^2 + y^2 + 6z^2$ if and only if $m \neq 9^k(9\ell + 3)$.

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- We wish to examine the representation of an integer m by a form Q , where Q has determinant D .
- We begin by introducing a ternary quadratic form $f(x, y, z)$ with determinant D given by:

$$mf(x, y, z) = (Ax + By + mz)^2 + (ax^2 + 2hxy + by^2)$$

- Note that by construction $f(x, y, z)$ represents m .
- Instead of using Gauss's result, we examine all forms of determinant D with all even cross terms.
- We identify an equivalence class that is not represented by all other forms with the same determinant but is represented by Q .
- $f(x, y, z)$ has determinant $D \Leftrightarrow ab - h^2 = Dm$.

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- Now we show how to select the constants A, B, a, b and h .
- First, we need the coefficients of $mf(x, y, z)$ to be divisible by m .
- We need the following equivalences to hold:

$$A^2 + a \equiv 0, B^2 + b \equiv 0, 2AB + 2h \equiv 0 \pmod{m}$$

- The last two equivalences hold if we set $B \equiv b \equiv h \equiv 0 \pmod{m}$.

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- So we still need to satisfy $A^2 + a \equiv 0 \pmod{m}$ and $ab - h^2 = Dm$.
- In addition we construct $\frac{A^2+a}{m}$ to represent the equivalence class we identified earlier.
- If we can achieve this, we have shown that $f(x, y, z)$ represents m and some element of the equivalence class not represented by the non- Q forms of determinant D .
- Thus $f(x, y, z) \sim Q$.

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- Next we show that it is possible to choose an a and A fulfilling $A^2 + a \equiv 0 \pmod{m}$ and $ab - h^2 = Dm$.
- Note that $-a \equiv A^2 \pmod{m}$ and $-Dm \equiv h^2 \pmod{a}$.
- Let a be a prime, $a \nmid m$, such that $\left(\frac{-a}{p}\right) = 1$ for all primes p dividing m .
- Hence $\left(\frac{-a}{m}\right) = 1$ by properties of Jacobi symbol.
- Now we need to check that $\left(\frac{-a}{m}\right) = 1$ and $\left(\frac{-Dm}{a}\right) = 1$ are guaranteed to hold simultaneously. We will put certain restrictions on a so these are guaranteed to hold.

Proof Outline for $x^2 + 2y^2 + 4z^2$

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Theorem

The quadratic form $x^2 + 2y^2 + 4z^2$ represents all integers not of the form $4^k(16\ell + 14)$ with $k, \ell \in \mathbb{Z}$.

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Positive definite quadratic forms with determinant 8 and even cross terms:

- $Q_1 : x^2 + 2y^2 + 4z^2$
- $Q_2 : x^2 + y^2 + 8z^2$
- $Q_3 : x^2 + 3y^2 + 3z^2 + 2yz$
- $Q_4 : 2x^2 + 2y^2 + 3z^2 + 2yz + 2xz$

How can we tell these forms apart?

Claim 1: If $m \equiv 6 \pmod{16}$, then m is not represented by Q_2 , Q_3 , or Q_4 .

Claim 2: Let $m \in \mathbb{Z}$. If $m \equiv 14 \pmod{16}$, then m is not represented by Q_1 .

Claim 3: If Q_1 represents $4m$, then $m \not\equiv 14 \pmod{16}$.

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Let $m \in \mathbb{Z}$ and not of the form $4^k(16l + 14)$. Consider the ternary quadratic form $f(x, y, z)$ of determinant $D = 8$ given by

$$mf(x, y, z) = (Ax + By + mz)^2 + (ax^2 + 2hxy + by^2)$$

with the determinant condition

$$ab - h^2 = 8m$$

and integers A, B such that $f(x, y, z)$ has integer coefficients and even cross terms. Note that $f(x, y, z)$ represents m .

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Choosing constants $A, B, a, b,$ and h :

Let a be an odd prime with $a \nmid m$ so that $\left(\frac{-8m}{a}\right) = 1$.

Consider

$$A^2 + a \equiv 0$$

$$B^2 + b \equiv 0$$

$$2AB + 2h \equiv 0 \pmod{m}.$$

The second and third congruence conditions are satisfied if we choose $B \equiv b \equiv h \equiv 0 \pmod{m}$. The first condition is met if we choose

$$\left(\frac{-a}{p}\right) = 1 \quad \forall p|m.$$

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Now we must check the consistency of

$$\left(\frac{-8m}{a}\right) = \left(\frac{-a}{m}\right) = 1.$$

In addition, we would like to construct each case so that

$$\frac{A^2 + a}{m} \equiv 6 \pmod{16}.$$

We have 11 congruences to check:

$$m \equiv 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 15 \pmod{16}.$$

Case $m \equiv 1, 9 \pmod{16}$

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Let $a = 2a_1$ where a_1 is an odd prime, $a_1 \nmid m$, so $\left(\frac{-8m}{a_1}\right) = 1$.
Additionally let $a_1 \equiv 3 \pmod{16}$. Now we must examine the
consistency of $\left(\frac{-a}{m}\right) = 1$ and $\left(\frac{-8m}{a_1}\right) = 1$:

$$\begin{aligned} 1 &= \left(\frac{-8m}{a_1}\right) = \left(\frac{-1}{a_1}\right) \left(\frac{2}{a_1}\right) \left(\frac{m}{a_1}\right) = \prod_{p|m} \left(\frac{p}{a_1}\right) \\ &= \prod_{p|m} \left(\frac{a_1}{p}\right) = \prod_{p|m} \left(\frac{-a_1}{p}\right) = \left(\frac{-2a_1}{m}\right) = \left(\frac{-a}{m}\right) \end{aligned}$$

as needed.

Case $m \equiv 1, 9 \pmod{16}$

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Now consider A .

Since $a = 2a_1$ and $a_1 \equiv 3 \pmod{16}$, we have $a \equiv 6 \pmod{16}$.

$$\frac{A^2+a}{m} \equiv 6 \pmod{16}$$

$$A^2 + a \equiv 6 \pmod{16}$$

$$A^2 \equiv 0 \pmod{16}$$

So choose $A \equiv 0 \pmod{4}$.

Numerical Example Using the Form $x^2 + 2y^2 + 4z^2$

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Example: Let $m = 17$.

Take $a = 38$, $b = 34$, $h = 34$, $B = 17$, $A = 8$.

So $17f(x, y, z) = (8x + 17y + 17z)^2 + (38x^2 + 68xy + 34y^2)$.

Dividing by m and using a change of variables, we arrive at our desired form:

$$\begin{aligned} f(x, y, z) &= 6x^2 + 19y^2 + 17z^2 + 20xy + 16xz + 34yz \\ &= (y + z)^2 + 2(x + y)^2 + 4(x + 2y + 2z)^2 \\ &\sim X^2 + 2Y^2 + 4Z^2 \end{aligned}$$

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Forms within the same genus:

- It is very difficult to distinguish two forms within the same genus as they represent the same equivalence classes.
- For instance, $x^2 + y^2 + 16z^2$ and $2x^2 + 2y^2 + 5z^2 + 2yz + 2xz$ are in the same genus. This makes it difficult to use our current method on $x^2 + y^2 + 16z^2$.

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Odd cross terms:

- The current method requires that all cross terms of the form have even coefficients. Another extension would be to alter the approach so that one could look at various forms with odd cross terms.

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Questions?

Thank you for your attention!