Classically Integral Quaternary Quadratic Forms Excepting at Most Two Values

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The Wake/Davidson Experience in Number Theory Research

July 28, 2016

Outline:

- Introduction to quadratic forms and (almost) universality
- Main results
- Tools for proving almost universality
    - Escalation
    - Modular forms
- Future Work

Definition

We say that a ***n*-ary integral positive definite quadratic form** is a homogeneous quadratic polynomial

$$Q(\vec{x}) = Q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j \in \mathbb{Z}[x_1, \ldots, x_n]$$

such that for all $\vec{x} \in \mathbb{Z}^n$, $Q(\vec{x}) \geq 0$ and $Q(\vec{x}) = 0$ if and only if $\vec{x} = \vec{0}$.

We will discuss forms where $n = 4$, i.e. quaternary forms.

Definitions

We can represent our quaternary quadratic form by a symmetric matrix $M_Q = (m_{ij}) \in M_4(\mathbb{Q})$ where $m_{ii} = a_{ii}$ and $m_{ij} = \frac{a_{ij}}{2}$ for $i \neq j$. Then, we have

$$Q(\vec{x}) = \vec{x}^{\,t} M_Q \vec{x}.$$

A form is **classically integral** if $M_Q \in M_4(\mathbb{Z})$ (all cross-terms are even).

Last, we say that two forms Q, Q' are **equivalent** if there exists $A \in GL_4(\mathbb{Z})$ such that $M_Q = A^t M_{Q'} A$.

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We call a form **universal** if for all $d \in \mathbb{N}$ there exists $\vec{x} \in \mathbb{Z}^4$ such that $Q(\vec{x}) = d$.

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The 15 Theorem

Barowsky, Damron, Mejia, Saia, Schock, Thompson

Theorem (Conway-Schneeberger 1993)

A classically integral quadratic form is universal if and only if it represents 1 *through* 15*. Furthermore, up to equivalence, there are* 204 *such forms in four variables.*

The 290 Theorem

Theorem (Bhargava-Hanke 2005)

A quadratic form is universal if and only if it represents 1 *through* 290*. Furthermore, up to equivalence, there are* 6436 *such forms in four variables.*

Almost Universality

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What about **almost universal** forms, ones which fail to represent finitely many natural numbers?

- In 1938 Halmos published a list of 88 quaternary diagonal forms excepting only one number
  - E.g. $x^2 + 2y^2 + 7z^2 + 13w^2$, which excepts only 5
  - (Though we found that two of these except *two* numbers)
- Can we extend the result to classically integral forms? What about those that except two numbers?
  - Yes!

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Main Result

Theorem (BDMSST 2016)

There are exactly 65 *sets* $S = \{m, n\}$ *(with* $m < n$*) for which there exists a quaternary classically integral positive definite quadratic form that represents precisely* $\mathbb{N} - S$*.*

Results

Table: Possible Pairs $\{m, n\}$ of Exceptions

m	n
1	$2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14$ $15, 17, 19, 21, 23, 25, 26, 30, 41$
2	$3, 5, 6, 8, 10, 11, 15, 18, 22, 30, 38$
3	$6, 7, 11, 12, 19, 21, 27, 30, 35, 39$
5	$7, 10, 13, 14, 20, 21, 29, 30, 35$
6	15
7	$10, 15, 23, 28, 31, 39, 55$
10	$15, 26, 40, 58$
14	$30, 56, 78$

Q: How did we find these pairs and corresponding forms?

A: Escalation!

Escalation

We define the **truant** of a quadratic form Q to be the smallest $t \in \mathbb{N} - \{m, n\}$ that Q does not represent.

In previous work, universal form candidates are systematically generated by taking an $(n-1)$-dimensional form Q with truant t and **escalating** to an n-dimensional form

$$Q' = Q + tx_n^2 + [\text{cross-terms involving } x_n].$$

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Escalation

Since $m \leq 15$, what are the possible n values?

- Fix m and escalate x^2 (or $2x^2$ if $m = 1$) until we get a four-variable form.
- Eliminate equivalent forms, those that miss more than 2 numbers (up to 10,000), and any that represent m .
- Find largest truant of any form on list.
  - This is our n_{max}.
- Run the escalation process from the beginning for m and each n with $m < n \leq n_{max}$.

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Now we have our candidates!

Q: How did we prove each candidate's almost universality?

A: Modular forms!

Definitions

Define the **representation number** of $k \in \mathbb{N}$ by Q as

$$r_Q(k) := \#\{\vec{x} \in \mathbb{Z}^4 : Q(\vec{x}) = k\}.$$

We then define the **theta series** associated to Q as

$$\Theta_Q(z) = 1 + \sum_{k \geq 1} r_Q(k) q^k$$

where $q = e^{2\pi i z}$.

$\Theta_Q(z)$ is a modular form of weight 2 and level N_Q with associated quadratic character χ_Q.

Thus we can decompose $\Theta_Q(z) = E_Q(z) + C_Q(z)$ where $E_Q(z)$ is Eisenstein and $C_Q(z)$ is a cusp form.

So we can write

$$r_Q(k) = a_E(k) + a_C(k)$$

for each coefficient in the Fourier expansion of Θ_Q.

Idea: If we can find K so that $k > K$ implies $r_Q(k) > 0$, then we will have only finite $k \leq K$ to check.

We do this by finding a lower bound for $a_E(k)$ and an upper bound for $|a_C(k)|$.

For each quaternary quadratic form Q, there are constants C_E (highly technical) and C_f (coming from the linear combination of $a_C(k)$ by Hecke eigenforms).

Theorem (Hanke 2004)

If $k \in \mathbb{N}$ is locally represented and satisfies

$$B(k) := \frac{\sqrt{k'}}{\tau(k)} \prod_{\substack{p|k, p\nmid N_Q, \\ \chi_Q(p)=-1}} \frac{p-1}{p+1} > \frac{C_f}{C_E},$$

where $\tau(k)$ is the number of positive divisors of k and k' is a divisor of k depending on the anisotropic primes of Q, then Q represents k.

Eligible Numbers

We say $k \in \mathbb{N}$ is an **eligible number** if it is locally represented and satisfies

$$B(k) \leq \frac{C_f}{C_E} \tag{1}$$

- i.e. "eligible" to be missed by Q
- $B(k)$ is multiplicative, so as k becomes divisible by more primes, $B(k)$ increases. Hence, finitely many k satisfy (1).

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We generate all eligible numbers via **eligible primes** (from a different inequality).

Checking Eligible Numbers

We're almost there! The last thing to do is check that $r_Q(k) > 0$ for all eligible $k \notin \{m, n\}$.

We could ask Magma for the $\Theta_Q(z)$ coefficients up to the maximum eligible number.

Unfortunately, this maximum can be *huge* (in the trillions).

- The form below excepts $\{1, 2\}$ and has a maximum eligible number of 20,561,800,549,290:

$$3x^2 - 4xy + 4y^2 - 2xz + 5z^2 - 4xw + 2zw + 7w^2.$$

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We have two tricks to save memory and time:

- Split local cover
- Approximate Boolean theta function

Split Local Cover (SLC)

To reduce the number of coefficients of $\Theta_Q(z)$ we must check, we use a split local cover of the form

$$Q' = dx^2 \oplus T,$$

where Q' is a quaternary form locally representing the same numbers as Q, T is a ternary subform of Q', and $d \in \mathbb{N}$.

For each eligible number k, we check for some $x \in \mathbb{N} \cup \{0\}$ such that $k - dx^2$ is represented by the ternary subform T.

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Approximate Boolean Theta Function (ABTF)

For a chosen precision Y, we store only a single bit for each number up to Y indicating whether it is represented by T.

Rather than computing the entirety of the theta series, we look only at vectors in the intersection of an appropriately chosen small rectangular prism with the ellipsoid $T(\vec{x}) \leq Y$.

Improvements:

- SLC & ABTF: stores $\sqrt{X}$ bits, $O\left(X^{1/4}\right)$ time
- Naive method: stores X bits, $O\left(X^2\right)$ time
- The following form ran in 31 minutes using the SLC & ABTF, compared to 4.42 hours without:

$$3x^2-2xy+4y^2-4xz-2yz+6z^2-2xw+8yw-2zw+7z^2.$$

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Example: $< 1, 2, 7, 13 >$

Halmos conjectured that the diagonal form

$$x^2 + 2y^2 + 7z^2 + 13w^2$$

represents all positive integers except for 5, but he could not provide a proof.

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Example: $< 1, 2, 7, 13 >$

This form has level $N_Q = 728$ and character $\chi_Q(p) = \left(\frac{182}{p}\right)$.

The dimension of the cuspidal subspace is 108.

Example: $< 1, 2, 7, 13 >$

We find that the bounding constants for this form are

$$C_f \approx 13.4964 \quad , \quad C_E = \frac{36}{71}$$

and compute:

- Number of eligible primes: 5,634
- Number of eligible numbers: 343,231
- Largest eligible number: 18,047,039,010

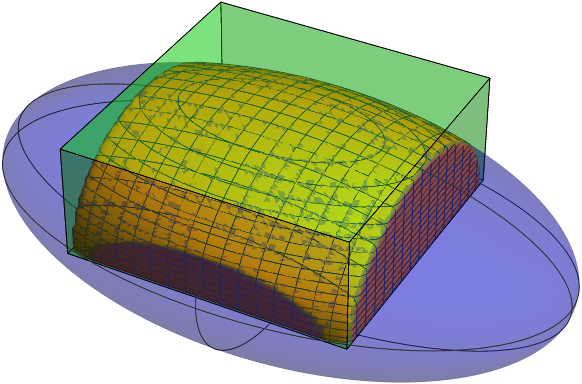
Example: $< 1, 2, 7, 13 >$

To check representation of eligible numbers, we use the split local cover $Q = x^2 \oplus T$ where

$$T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}.$$

For our approximation, we look only at vectors in the intersection of a chosen prism with the ellipsoid $T(\vec{x}) \leq Y$ for chosen precision Y.

Example: $< 1, 2, 7, 13 >$



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- Employing the ABTF with our chosen SLC to check all eligible numbers, we find that all natural numbers aside from 5 are indeed represented.
- These computations took approximately 4 minutes and 30 seconds.
- This proves Halmos' conjecture.

Future Work

- Higher-dimensional forms excepting $\{m, n\}$
  - Currently working on this.
  - Still able to use quaternary tools.
- Quadratic forms with odd cross-terms.
  - Initial work has shown that it is computationally taxing to generate forms and eliminate equivalent ones.

Acknowledgements

We'd like to thank the NSF for the grant supporting our research, Kate Thompson and Jeremy Rouse for their invaluable guidance and contributions (and for running the REU!), and Wake Forest University for opening their campus and servers to us.

Additional thanks to UGA and the conference organizers.

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